MATRIX ALGEBRA

(1) What is a matrix?

An $\underset{\substack{\uparrow \\ \text{row}}}{m} \times \underset{\substack{\uparrow \\ \text{column}}}{n}$ array of (complex) numbers.

e.g.

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \qquad a_{\underset{\substack{\uparrow \\ \text{row\#}}}{i}\underset{\substack{\uparrow \\ \text{column\#}}}{j}}$$

(2) Rules:

Addition: $\underline{\underline{C}} = \underline{\underline{A}} + \underline{\underline{B}} \qquad c_{ij} = a_{ij} + b_{ij}$

Multiplication: $\underline{\underline{C}} = \underline{\underline{A}}\,\underline{\underline{B}} \qquad c_{ij} = \sum_k a_{ik} b_{kj}$

e.g.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + 2\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+10 & 2+12 \\ 3+14 & 4+16 \end{pmatrix} = \begin{pmatrix} 11 & 14 \\ 17 & 20 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1\times5+2\times7 & 1\times6+2\times8 \\ 3\times5+4\times7 & 3\times6+4\times8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

Distribution: $\underline{\underline{A}}(\underline{\underline{B}} + \underline{\underline{C}}) = \underline{\underline{A}}\,\underline{\underline{B}} + \underline{\underline{A}}\,\underline{\underline{C}}$

Association: $\underline{\underline{A}}(\underline{\underline{B}}\,\underline{\underline{C}}) = (\underline{\underline{A}}\,\underline{\underline{B}})\underline{\underline{C}}$

Commutation: $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$

Trace: $\mathrm{Tr}(\underline{\underline{A}}) = \sum_k A_{kk}$

e.g. $\mathrm{Tr}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 5$

Determinant: $|\underline{\underline{A}}\,\underline{\underline{B}}| = |\underline{\underline{A}}|\,|\underline{\underline{B}}| \qquad |\underline{\underline{A}}| = \sum_j (-1)^{i+j} a_{ij} |A_{ij}|$

e.g. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\times4 - 2\times3 = -2$

Determinant:

$$|\underline{\underline{A}}| = \sum_j (-1)^{i+j} a_{ij} |A_{ij}|$$

e.g. $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21} (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$+ a_{31} (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11} a_{22} a_{33} + a_{21} a_{32} a_{13} + a_{31} a_{12} a_{23}$$
$$- a_{11} a_{32} a_{23} - a_{21} a_{12} a_{33} - a_{31} a_{22} a_{13}$$

e.g. $$\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 3 & 6 & 4 \end{vmatrix} = (1)(-1)(4) + (2)(3)(3) + (2)(6)(1)$$
$$-(3)(-1)(1) - (1)(6)(3) - (4)(2)(2)$$

$$= -5$$

e.g. $$\begin{vmatrix} a & b & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 & 0 \\ 0 & 0 & 0 & f & g & h \\ 0 & 0 & 0 & i & j & k \\ 0 & 0 & 0 & l & m & n \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} |e| \begin{vmatrix} f & g & h \\ i & j & k \\ l & m & n \end{vmatrix}$$

Inverse: $\underline{\underline{A}}^{-1}$ $\qquad \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}$ $\qquad \underline{\underline{I}} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

Transpose: $\tilde{\underline{\underline{A}}}$ $\qquad \tilde{a}_{ij} = a_{ji}$

e.g. $$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \underline{\underline{A}} \qquad \tilde{\underline{\underline{A}}} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

Hermitean adjoint : $\underline{\underline{A}}^{+} = \tilde{\underline{\underline{A}}}^{*}$ $\qquad (a^{+})_{ij} = a_{ji}^{*}$

Hermitean matrix : $\underline{\underline{H}}^{+} = \underline{\underline{H}}$

Unitary matrix : $\underline{\underline{U}}^{+} = \underline{\underline{U}}^{-1}$

(3) Unitary Transformation (similarity):

$$\underline{\underline{A}}' = \underline{\underline{S}}\,\underline{\underline{A}}\,\underline{\underline{S}}^{-1} \quad \text{where } \underline{\underline{S}} \text{ is unitary.}$$

(4) Theorems:

(1) Hermitean matrix has "real" eigenvalues.

(2) A hermitean matrix can be diagonalized by a unitary transformation, i.e. There is a matrix $\underline{\underline{S}}$ such that

$$\underline{\underline{S}}\,\underline{\underline{H}}\,\underline{\underline{S}}^{-1} = \underline{\underline{D}} \quad \text{(diagonal matrix)}.$$

(3) $|\underline{\underline{S}}\,\underline{\underline{H}}\,\underline{\underline{S}}^{-1}| = |\underline{\underline{D}}|$

In other words, the eigenvalues of $\underline{\underline{H}}$, $\underline{\underline{S}}\,\underline{\underline{D}}\,\underline{\underline{S}}^{-1}$ are equal, i.e. physical content unaffected by unitary transformation.

(4) If two hermitean matrices $\underline{\underline{H}}_1, \underline{\underline{H}}_2$ commute, then they can be simultaneously diagonalized.
i.e. there is a $\underline{\underline{S}}$ such that

$$\underline{\underline{S}}\,\underline{\underline{H}}_1\,\underline{\underline{S}}^{-1} = \underline{\underline{D}}_1$$

$$\underline{\underline{S}}\,\underline{\underline{H}}_2\,\underline{\underline{S}}^{-1} = \underline{\underline{D}}_2$$

Proof of converse:

$$\begin{array}{c} S H_1 S^{-1} = D_1 \\ S H_2 S^{-1} = D_2 \\ \hline S H_1 \underbrace{S^{-1} S} H_2 S^{-1} = D_1 D_2 \end{array}$$

$$S H_1 H_2 S^{-1} = D_1 D_2$$

Similarly, $S H_2 H_1 S^{-1} = D_2 D_1$

$$S [H_1 H_2 - H_2 H_1] S^{-1} = 0$$

diagonal matrices commute

e.g. To find the eigenvector (+ eigenvalues) of matrix $\underline{\underline{A}} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$

(1) Eigenvalues: $|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = \begin{vmatrix} -\lambda & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\lambda & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\lambda & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & -\lambda \end{vmatrix} = 0$

$$-2\lambda(-8\lambda^3 + 4\lambda) - 4\lambda^2 + (-4\lambda^2) = 0$$

$$16\lambda^4 - 16\lambda^2 = 0$$

$$\lambda^2(\lambda^2 - 1) = 0 \Rightarrow \begin{array}{l} \lambda = 0 \text{ (twice)} \\ \lambda = \pm 1 \end{array}$$

(2) Eigenvectors: $\underline{\underline{A}}\,\underline{v} = \lambda\,\underline{v}$

Let $\lambda = 0$ and $\underline{v} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$ then $\frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0$

$$\frac{1}{2}(c_2 + c_3) = 0 \quad \therefore c_2 = -c_3$$

$$\frac{1}{2}(c_1 + c_4) = 0 \quad \therefore c_1 = -c_4$$

$\underline{v}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -1/2 \\ 1/2 \end{pmatrix}$ are two possible solutions.